In general for derivatives of any order k

$$\frac{\partial^k u}{\partial t^m \partial x^r}, \quad m+r=k, \quad 0 \leqslant m, \quad r \leqslant k \leqslant l-2$$

estimates of the error $O(\varepsilon^d)$ are similar to (4.6), (4.9) with the exponent $d = d(k) = [1 - (k+1)\,l^{-1}]\,(p+1) > 0$.

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THE TECHNICAL STABILITY OF PARAMETRICALLY EXCITABLE DISTRIBUTED PROCESSES*

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The technical stability /1, 2/ - in a finite interval of time - of parametrically excitable processes with distributed parameters, i.e. processes described by partial differential equations with time-dependent (particularly time-periodic) coefficients, is investigated. Using the comparison method /3—6/ in conjunction with Lyapunov's second method /7/, the sufficient conditions for technical stability /1—6/ with respect to a specified measure are obtained. The determination of the corresponding differential inequalities of the comparison /4/ rests on the extremal properties of Rayleigh's ratios for selfadjoint operators in Hilbert space /8—12/. This approach is connected with the solution of the eigenvalue problem. The results obtained are used to establish the sufficient conditions using the specified measure in the problem of a clamped support /9/ loaded with some longitudinal force, particularly one which is time-periodic. At the same time the domain of technical stability is connected with the small parameter and the conditions of positive definiteness of Lyapunov's functional and the boundedness of the corresponding eigenvalues /11, 13, 14/. The technical stability of distributed-parameter systems for constantly acting perturbations have been investigated previously /1/, and the technical stability of processes with after-effect was examined using an axiomatic approach /2/. The problem of the technical stability of some systems which simultaneously contain distributed and lumped parameters was considered in /15/.

1. A theorem on the technical stability of parametrically excitable processes. Consider a class of dynamic processes in the domain $D \subset R^\nu$ with boundary C, where R^ν is a ν-dimensional Euclidean space with the vector of coordinates $x = (x_1, \ldots, x_\nu)$,

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described in a finite interval of time $T_1 \subset T = [t_0, +\infty)$ by an equation with boundary and initial conditions

$$\partial u(t, x)/\partial t = L(t) u(t, x), \quad x \in D, \quad t \in T_1 \tag{1.1}$$

$$Gu(t, x) = 0, \quad x \in C \tag{1.2}$$

$$u(t_0, x) = u_0(x), \quad t_0 \in T_1 \tag{1.3}$$

Here $u(t, x)$ is a $2N$-dimensional vector of state, $u_0(x)$ is a function which has all partial derivatives of the necessary order in the domain $D \subset R^\nu$, and $L(t)$ is a $2N \times 2N$ matrix of linear differential operators in partial derivatives with respect to the spatial variable, with time-dependent continuous coefficients (in particular, those that satisfy some conditions of periodicity). The parametrically excitable processes with distributed parameters are described using systems (1.1)–(1.3) G is a linear differential operator with respect to the spatial variable that is not time-dependent. We formulate the following problem: it is required to investigate the technical stability of the state $u(t, x) \equiv 0$ of process (1.1)–(1.3).

We introduce into consideration the real functional space H of the $2N$-dimensional vectors of the continuous functions, defined in $T_1 \times D$. For each pair of vectors $z_1, z_2 \in H$ we shall determine the scalar product

$$(z_1, z_2) = \int_D z_1'(t, x) z_2(t, x) dx \tag{1.4}$$

(the prime denotes transposition). Henceforth, we will assume that the space H is extended in a way that it is Hilbertian /8, 12/. We shall denote the norm in it, induced using the scalar product (1.4), by $\|\cdot\|$.

Suppose the set W of states of the process is a subset of the set H, the elements of which satisfy the boundary conditions (1.2) and other conditions of smoothness guaranteeing the continuity of the function $L(t) u$ in $T_1 \times D$ for the operator $L(t)$ that acts in the domain W. Then we can compare the element of the set W with each solution $u(t, x)$ of problem (1.1)–(1.3) for each instant of time t, the solution $u(t, x)$ itself forms some trajectory in W, and $L(t)$ is an operator in H which acts in the domain $W \subset H$, which we shall write as $L(t)$: $W \to H$.

Consider problem (1.1)–(1.3) in the domain

$$\Omega = \{t, x, u\colon t \in T_1, \; x \in D \subset R^\nu, \; \|u\| \leqslant a = \text{const} > 0, \\ \forall u \in W \subset H, \; L(t)\colon \; W \to H\}$$

For each pair $u, v \in W$ we shall determine the measure characterizing the degree of closeness between the two states

$$\rho(u, v) = \left[(A_0 z, A_0 z) + \sum_{i=1}^{n} (A_i z, A_i z)\right]^{1/2}, \quad z = u - v$$

where $A_0 = E$ is the identity transformation operator, and $A_1, A_2, \ldots, A_n$ are linear operators in H of order $j = 1, 2, \ldots, n$ respectively, that act in the domain W, are not time-dependent and are differential with respect to the space variable. Obviously, we have

$$\rho(u, 0) = \left[(u, u) + \sum_{i=1}^{n} (u, A_i^* A_i u)\right]^{1/2} = (u, Mu)^{1/2}, \quad M = E + \sum_{i=1}^{n} A_i^* A_i \tag{1.5}$$

Integration by parts is carried out using boundary conditions (1.2), by virtue of which the boundary values that emerge equal zero; A_i^* signifies the operator which is adjoint to the operator A_i /8, 12/.

We can verify that $\rho(u, v)$ satisfies all the axioms of the measure of the processes in the general case /1, 2, 16, 17/.

Consider the functional

$$V[u, t] = (u, B(t) u) \tag{1.6}$$

The operator $B(t)$: $H \to H$ can generally contain the necessary differential elements which are linear with respect to the space variable, like the matrix has the same dimensions as $L(t)$, and is a selfadjoint operator in the following sense:

$$(v, B(t) w) = (w, B(t) v), \quad v, w \in W \tag{1.7}$$

Therefore the functional V is explicitly time-dependent thanks to the time-dependent operator $B(t)$: $H \to H$. We also assume that $B(t)$ contains coefficients for which $B(t)$ is t-differentiable. Suppose the operator $B^{-1}(t)$, inverse to $B(t)$ exists. In addition, we shall specify the operator $B(t)$ in such a way that the functional $V[u, t]$ is positive definite with respect to the measure $\rho(u, 0)$, i.e.

$$V[u, t] \geqslant \alpha \rho^2(u, 0), \quad \forall t \in T_1 \cap T_2 \tag{1.8}$$

for some constant $\alpha > 0$. Obviously, by virtue of (1.5) condition (1.8) has the form

$$(u, [B(t) - \alpha M] u) \geqslant 0, \quad \alpha > 0 \tag{1.9}$$

Definition 1. The unperturbed process $u(t, x) \equiv 0$, corresponding to the boundary value problem (1.1)–(1.3), is termed technically stable in the finite interval $T_1 \subset T$ with respect to the specified measure $\rho(u, 0)$, if, along the perturbed solution $u(t, x)$ of problem (1.1)–(1.3) for the positive definite functional $V[u, t]$ with respect to the measure $\rho(u, 0)$ for the specified operator $B(t)$: $H \to H$ the condition $V[u(t, x), t] \leqslant P(t)$, $t \in T_1$ holds as soon as $V[u(t_0, x), t_0] \leqslant b$, where the function $P(t)$ defined in the interval T_1 satisfies the condition

$$P(t_0) \geqslant b, \quad \bar{P} = \sup_{t \in T_1} \{P(t)\} < +\infty, \quad b = \text{const} > 0$$

At the same time the function $P(t)$ and the constants b and T_1 are specified in advance.

We will assume that the operator $L^*(t)$, which is adjoint to the operator $L(t)$ is known. Suppose the operator $N(t)$: $W \to W$ equals $N(t) = L^*(t) B(t) + L(t) B(t) + B^{\cdot}(t)$, where $B^{\cdot}(t) = dB/dt$, and we have the eigenvalue problem

$$N(t) u = \lambda B(t) u, \quad u \in W, \quad t \in T_1 \tag{1.10}$$

for the specified boundary conditions (1.2), the eigenvalues of which are real and bounded quantities $\{\lambda_n(t)\}$ for all $t \in T_1$ /18/.

We shall use $\lambda_{\max}(t)$ to denote the maximum eigenvalue of problem (1.10).

Consider in the domain

$$\Lambda = \{t, y, \lambda_{\max}(t): t \in T_1 \subset T, \quad -\infty < y < +\infty,$$
$$|\lambda_{\max}(t)| \leqslant k, \quad k = \text{const} > 0\}$$

the function $\Phi(t, y, \lambda_{\max}(t))$, which is continuous in Λ and vanishes when $y = 0$, and also the corresponding scalar Cauchy problem

$$dy/dt = \Phi(t, y, \lambda_{\max}(t)), \quad y(t_0) = y_0 \tag{1.11}$$

Definition 2. The solution $\bar{y}(t) = \bar{y}(t, t_0, y_0, \lambda_{\max})$ of problem (1.11), emerging from the point $(t_0, y_0) \in \Lambda$ and existing in the interval $T_2 \subset T$ which contains within it the point t_0, will be termed upper (y-upper), if the inequality $\varphi(t) \leqslant \bar{y}(t)$, $t \in T_2$ holds for all $t \in T_2 \subset T$, bounded by the value $\lambda_{\max}(t)$ for any other solution $\varphi(t) = \varphi(t, t_0, y_0, \lambda_{\max}(t))$ of problem (1.11) which emerges from the point $(t_0, y_0) \in \Lambda$ and is determined in $T_2 \subset T$.

Definition 3. Eq.(1.11) is called a generalized comparison equation for the above initial condition for the class of processes (1.1)–(1.3), if the inequality $V[u(t, x), t] \leqslant \bar{y}(t)$ holds in the combined interval of time $T_1 \cap T_2$ of the existence of the solution $u(t, x)$ of problem (1.1)–(1.3) and of the y-upper solution $\bar{y}(t, t_0, y_0, \lambda_{\max})$ of problem (1.11) along the solution $u(t, x)$ of problem (1.1)–(1.3), satisfying for each $t \in T_1$ and finite $\lambda_{\max}(t)$ the condition $u(t, x) \in W \subset H$.

Theorem. We will assume that

1) the differential operator $L(t)$ of problem (1.1)–(1.3) for the specified properties of regularity of its coefficients is an operator in H, which acts in the domain $W \subset H$;

2) the functional $V[u, t] = (u, B(t) u)$, - which is positive definite with respect to the measure $\rho(u, 0)$ - exists, where $B(t)$: $H \to H$ is a selfadjoint operator in the sense of definition (1.7), satisfying the necessary properties of differentiability with respect to t;

3) the operator $L(t)$ satisfies the following conditions: a) for $L(t)$ the adjoint operator $L^*(t)$ exists, such that for all $v, w \in W$ the equation $(L(t) v, B(t) w) = (v, L^*(t) B(t) w)$ holds; b) in the eigenvalue problem (1.10) the operator $B^{-1}(t) N(t)$, where $B^{-1}(t)$ is inverse to the operator $B(t)$, is compact /12/ for all $t \in T_2 \subset T$ ($t_0 \in T_2$, $t_0 \in T_1$, $T_1 \subseteq T_2$).

4) the upper solution $\bar{y}(t) = \bar{y}(t, t_0, y_0, \lambda_{\max}(t))$ of the scalar comparison-type problem (1.11), corresponding to the maximum eigenvalue $\lambda_{\max}(t)$ of the eigenvalue problem (1.10), satisfies the inequalities $\bar{y}(t) \leqslant P(t)$, $\bar{y}(t_0) \leqslant b$, $t, t_0 \in T_1 \cap T_2$.

Then the dynamic processes described by Eqs.(1.1) and boundary conditions (1.2), (1.3) in the Hilbert space H are technically stable using the measure $\rho(u, 0)$ in the domain Ω in a finite interval of time $T_1 \cap T_2$.

Proof. Bearing in mind conditions 1)-3) and formulating $V(t) = V[u(t, x), t]$, we shall calculate the time-derivative from the functional $V[u, t]$ along the solution $u(t, x) \in W \subset H$ of problem (1.1)–(1.3)

$$dV(t)/dt = (L(t) u, \ B(t) u) + (u, B(t) L(t) u) +$$
$$(u, B^{\cdot}(t) u) = (u, N(t) u)$$

Here

$$N(t) = L^*(t) B(t) + B(t) L(t) + B^{\cdot}(t) \tag{1.12}$$

Along the trajectories $v(t, x)$, $w(t, x) \in W$ we shall calculate the time-derivative from both sides of Eq. (1.7)

$$\frac{d}{dt}(v, B(t)w) = \left(\frac{dv}{dt}, B(t)w\right) + (v, B^{\cdot}(t)w) +$$
$$\left(v, B(t)\frac{dw}{dt}\right) = (v, L^{*}(t)B(t)w) + (v, B(t)L(t)w) + (v, B^{\cdot}(t)w) =$$
$$(v, N(t)w), \qquad \frac{d}{dt}(w, B(t)v) = (w, N(t)v)$$

Consequently, the operator $N(t)$ is selfadjoint in the sense of definition (1.7). According to condition 3b) of the theorem, problem (1.10) with the operator (1.12) for all $t \in T_2 \subset T$ has only finite real eigenvlaues, i.e. its maximum eigenvalue $\lambda_{\max}(t)$ is a bounded real quantity /8, 12/.

Bearing in mind the above, consider the ratio of the two scalar products $\lambda(u, t) = (u, N(t)u)/(u, B(t)u)$. It follows from the theorem on the extremal properties of the ratios $\lambda(u, t)$ that

$$\lambda(u, t) \leqslant \lambda_{\max}(t), \; t \in T_2 \subset T$$

Then we have the following estimate along the solution $u(t, x)$ of problem (1.1)–(1.3):

$$\frac{d}{dt} V[u(t, x), t] \leqslant \lambda_{\max}(t) V[u(t, x), t], \qquad t \in T_1 \tag{1.13}$$

Estimate (1.13) enables us to consider /3–6/ the following scalar Cauchy problem of the type (1.11), generated by the eigenvalue problem (1.10).

$$dy/dt = \lambda_{\max}(t)\, y, \;\; y(t_0) = y_0 \geqslant V[u(t_0, x), t_0] \;\; t \geqslant t_0, \tag{1.14}$$
$$t, t_0 \in T_2 \subset T$$

i.e. we shall take as the function

$$\Phi(t, V(t), \lambda_{\max}(t)) = \lambda_{\max}(t)\, V[u(t, x), t]$$

Using the upper solution of problem (1.14), which are equals

$$\bar{y}(t) = y_0 F(t), \qquad F(t) = \exp\left[\int_{t_0}^{t} \lambda_{\max}(\tau)\, d\tau\right]$$

according to the theorem on differential inequalities(/11/),Theorem 9.5), we have the following estimate along the solution $u(t, x)$ of boundary value problem (1.1)–(1.3) for the positive definite functional $V[u, t]$ using the measure $\rho(u, 0)$

$$V[u(t, x), t] \leqslant y_0 F(t) \leqslant P(t)$$

The right-hand side of this inequality follows from conditions 4) of the theorem.

In addition, bearing in mind the conditions of problem (1.14), we have the inequalities $V[u(t_0, x), t_0] \leqslant y_0 \leqslant b$ when $t = t_0$.

The function $P(t)$ and the constant b are specified in advance according to Definition 1. Consequently, the processes of (1.1)–(1.3) are technically stable in the domain Ω in the finite interval of time $T_1 \cap T_2$ using the measure $\rho(u, 0)$. The theorem is proved.

2. The technical stability of a support with clamped ends, loaded with a dynamic longitudinal force. The motion of the support is described using an equation with initial and boundary conditions /9, 19/

$$\frac{\partial^2 w}{\partial t^2} = -\frac{\partial^4 w}{\partial x^4} - f(t)\frac{\partial^2 w}{\partial x^2} - \beta\frac{\partial w}{\partial t} \tag{2.1}$$
$$w(t, x)|_{t=0} = w_0(x), \;\; \partial w(t, x)/\partial t|_{t=0} = v_0(x)$$
$$w(t, 0) = w(t, 1) = 0, \;\; \partial w(t, 0)/\partial x = \partial w(t, 1)/\partial x = 0$$

where $w_0(x)$, $v_0(x)$ are quadruply continuously differentiable functions with respect to $x \in D \equiv [0, 1]$. We assume, in particular, that the force $f(t)$ varies cosinusoidally with time

$$f(t) = 4\pi^2(R_0 + R_t \cos \omega t) \tag{2.2}$$
$$R_0 = \frac{P_0 l^2}{4\pi^2 EI}, \qquad R_t = \frac{P_t l^2}{4\pi^2 EI}$$

Here P_0 is the constant component of the compressible force, P_t is the constant of the amplitude of the pulsating component of the longitudinal force, l is the length of the girder, E is Young's modulus, I is the moment of inertia of the cross-section of the girder with respect to its axis, passing through its centre of mass, and β is the damping factor.

A different regular representation is also permissible for the force $f(t)$

We can write problem (2.1) in the form

$$\frac{\partial u(t,x)}{\partial t} = L(t)u(t,x); \quad u(t,x) = \left\| \begin{matrix} w(t,x) \\ v(t,x) \end{matrix} \right\|, \quad v \equiv \frac{\partial w}{\partial t} \tag{2.3}$$

$$L(t) = \left\| \begin{matrix} 0 & 1 \\ -\frac{\partial^4}{\partial x^4} - f(t)\frac{\partial^2}{\partial x^2} & -\beta \end{matrix} \right\| \tag{2.4}$$

$$u(t,x)|_{t=0} = u_0(x); \quad u_0(x) = \left\| \begin{matrix} w_0(x) \\ v_0(x) \end{matrix} \right\| \tag{2.5}$$

$$u(t,0) = u(t,1) = 0, \quad \frac{\partial u(t,0)}{\partial x} = \frac{\partial u(t,1)}{\partial x} = 0 \tag{2.6}$$

We shall denote the finite interval of time $T = [0, L\mu^{-1}]$, μ is a small positive parameter, and the constant $L > 0$ in general depends on the parameters of the system. Consider the real Hilbert space H of the vectors $u = \{u_1(t, x), u_2(t, x)\}$ with the continuous functions $u_1(t, x)$, $u_2(t, x)$ when $t \in T$, $x \in D$, for which the scalar product of each pair

$$(u, v) = \int\limits_D \sum_{i=1}^{2} u_i(t,x) v_i(t,x)\,dx, \quad u, v \in H$$

The operator $L(t)$ is an operator in H, which is effective in the domain $\overline{W}(T \times D)$, i.e. $L(t)\colon \overline{W} \to H$. The domain of definition $\overline{W}$ of the operator (2.4) is a set of two-component vectors $u = \{u_1(t, x), u_2(t, x)\} \in H$, the components of which have the partial derivatives with respect to x of the fourth-order inclusive which belong to the Hilbert space $L^2(D)$ for each $t \in T$ and which satisfy boundary conditions (2.6). Then the solution of problem (2.3)–(2.6) $u(t, x)$ for all $t \in T$ determines the trajectory in $\overline{W}$.

We shall take as the measure of $\rho(u, 0)$ in $\overline{W} \subset H$

$$\rho(u,0) = \left\{ \int\limits_0^1 \left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2 + w^2 + v^2 \right] dx \right\}^{1/2}$$

Hence, bearing in mind boundary conditions (2.6), we obtain the following representation for the operator M:

$$M = \left\| \begin{matrix} \frac{\partial^4}{\partial x^4} - \frac{\partial^2}{\partial x^2} + 1 & 0 \\ 0 & 1 \end{matrix} \right\| \tag{2.7}$$

The conjugate operator $L^*(t)$ in this case (bearing in mind boundary conditions (2.6)) has the form of a matrix transposed in matrix (2.4).

To use the comparison method based on an analysis of the eigenvalue problem corresponding to (2.3)–(2.6), consider the functional

$$V[u, t] = (u, B(t)u) \tag{2.8}$$

where the operator $B(t)$ is determined in the form of the following matrix:

$$B(t) = \left\| \begin{matrix} \frac{\partial^4}{\partial x^4} + \gamma(t)\frac{\partial^2}{\partial x^2} + \delta_+ & \frac{\beta}{4} \\ \beta/2 & 1 \end{matrix} \right\|, \quad \delta_\pm = \alpha_1 \pm \frac{\beta^2}{4} \tag{2.9}$$

($\gamma(t)$ is some cofficient which is continuously time-differentiable $t \in [0, L\mu^{-1}]$, and α is some constant parameter). The functional V is positive definite with respect to the measure $\rho(u, 0)$ with the conditions

$$0 < \mu < 1, \ (1-\mu)\,4\pi^2 - (\mu + \varepsilon) > \gamma(t) \tag{2.10}$$

$$\alpha_1 > \mu - \frac{\beta^2}{4} - \varepsilon\pi^2 + \frac{\beta^2}{8(1-\mu)}, \quad \varepsilon = \text{const} > 0$$

which follow from the inequality $(u, [B(t) - \mu M]u) \geqslant 0$, where the operator M is defined by (2.7). From (2.4) and (2.9) for $N(t) = L^*(t)B(t) + B(t)L(t) + B^{\cdot}(t)$ we obtain the representation

$$N(t) = \left\| \begin{matrix} -\beta\frac{\partial^4}{\partial x^4} - h(t) & -g(t)\frac{\partial^2}{\partial x^2} + \delta_- \\ -g(t)\frac{\partial^2}{\partial x^2} - \delta_+ & -\beta \end{matrix} \right\| \tag{2.11}$$

$$h(t) = \beta f(t) - \frac{d}{dt}\gamma(t), \quad g(t) = f(t) - \gamma(t)$$

The eigenvalue problem, which corresponds to Eqs.(2.9)–(2.11), is determined using an operator equation of the type (1.10), and the unknown eigenvector $y \in \overline{W}$ is a two-component vector: $y = (y_1, y_2)$. Using the exclusion method this system only reduces to an equation for one component, for example y_1:

$$\frac{\partial^4 y_1}{\partial x^4} + k(t)\frac{\partial^2 y_1}{\partial x^2} + c(t)\, y_1 = 0 \tag{2.12}$$

$$k(t) = \frac{(\lambda+\beta)\,[h(t)+\lambda\gamma(t)] + 2g(t)(\delta_- - \lambda\beta/2)}{(\lambda+\beta)^2 - g^2(t)} \tag{2.13}$$

$$c(t) = \frac{\lambda(\lambda+\beta)\,\delta_+ - (\delta_- - \lambda\beta/2)}{(\lambda+\beta)^2 - g^2(t)}$$

The function y_1 satisfies the zero boundary conditions

$$y_1(t,0) = y_1(t,1) = 0, \qquad \frac{\partial y_1}{\partial x}(t,0) = \frac{\partial y_1}{\partial x}(t,1) = 0 \tag{2.14}$$

Eq.(2.2) or a corresponding system of the (1.10) type has constant coefficients for fixed t. In this case, the characteristic equation which corresponds to (2.12) has the roots $\pm i$ $(k/2 \pm \sqrt{k^2/4 - c})^{1/2}$ (any set of signs). Let us consider further the case when the coefficient $k(t)$ is positive for all $t \in [0, L\mu^{-1}]$. For these values of k and t we obtain two series of eigenfunctions, namely:

a) when $c > 0$ and $k^2 > 4c$

$$y_1(t,\ x) = C_1(t)\ \sin r_1 x + C_2(t) \cos r_1 x + C_3(t)\ \sin r_2 x + C_4(t)\ \cos r_2 x;\ r_{1,2} = (k/2 \pm \sqrt{k^2/4 - c})^{1/2}$$

where the quantities r_1, r_2, as follows from (2.14), satisfy the condition

$$2r_1 r_2 (\cos r_1 \cos r_2 - 1) + (r_1^2 + r_2^2)\ \sin r_1 \sin r_2 = 0 \tag{2.15}$$

b) when $c < 0$ and $k^2 > 4c$

$$y_1(t,\ x) = C_1(t) \sin r_1 x + C_2(t) \cos r_1 x + C_3(t)\,\mathrm{sh}\, r_2 x + C_4(t)\,\mathrm{ch}\, r_2 x;\ r_{1,2} = (\pm k/2 + \sqrt{k^2/4 - c})^{1/2}$$

where the quantities r_1, r_2, according to Eq.(2.14), satisfy the condition

$$2r_1 r_2 (\cos r_1 \mathrm{ch}\, r_2 - 1) + (r_1^2 - r_2^2) \sin r_1 \mathrm{sh} r_2 = 0 \tag{2.16}$$

It follows from the conditions of this eigenvalue problem that an infinite discrete set of eigenvalues λ_n exists, for which the pairs of numbers (k_n, c_n), corresponding to the numbers λ_n, will satisfy conditions (2.15) or (2.16). For the maximum eigenvalue for each $t \in [0, L\mu^{-1}]$ we have

$$\lambda_{\max}(t) = \max_{k_n}\left\{-\beta - \frac{\gamma^{\cdot}(t)}{2[k_n - \gamma(t)]} + \left[\left(\frac{\gamma^{\cdot}(t)}{2[k_n-\gamma(t)]}\right)^2 + \frac{2g(t)\,\delta_+}{k_n - \gamma(t)} + \frac{g^2(t)}{1-\gamma(t)/k_n}\right]^{1/2}\right\}, \quad n = 1, 2, \ldots \tag{2.17}$$

The quantity (2.17) becomes unbounded when $k_n = \gamma(t)$ $(\gamma(t) > 0)$ for some integral n. In this case from (2.13) there follows the equation

$$c_n(\lambda+\beta)^2 - 4c_n\gamma^{-2}(t)\,\delta_+^2 = \alpha_1(\lambda+\beta)^2 - \delta_+^2$$

from which, of necessity, it follows that

$$\gamma^2(t) = 4c_n,\ \ c_n = \alpha_1 \tag{2.18}$$

for the above integral n. Hence also $k_n^2 = 4c_n$. Therefore the eigenvalue will be unbounded when the pair $(k_n, c_n) = (\gamma, \alpha_1)$ satisfies one of conditions (2.15) or (2.16). But this is not practicable for, as follows from (2.18), such a pair contradicts conditions a) and, moreover, conditions b). Consequently, the conditions of the initial problem, including the conditions of positive definiteness (2.10) of the functional (2.8), guarantee the boundedness of the eigenvalues of the problem for the operator (2.11). On the other hand, the conditions of the initial problem satisfy the conditions of the theorem proved in Sect.1.

Thus, for any integral n $k_n \neq \gamma(t)$ for all $t \in [0, L\mu^{-1}]$. Therefore all k_n $(n = 1, 2, \ldots)$, as follows from conditions (2.10), must satisfy the inequality

$$k_n > (1-\mu)\ 4\pi^2 - (\mu + \varepsilon),\ n = 1, 2, \ldots \tag{2.19}$$

for each $t \in [0, L\mu^{-1}]$ together with conditions (2.15) or (2.16). In addition, for any fixed $t \in [0, L\mu^{-1}]$ the quantities k_n can be arranged in the order in which they increase: $k_1 < k_2 < k_3 < \ldots$. Therefore for each fixed instant of time t, as follows from (2.17), maximum λ is attained either when $k = k_1$, or when $k = k_\infty$ as a function of the value of t. If $k_n \to \infty$, the quantity λ_n approaches the limit $\lambda_\infty = -\beta + g(t)$. Consequently, the quantity

$$Q = \sup_{t \in [0,\, L\mu^{-1}]} |\lambda_{\max}(t)|$$

which obviously depends on conditions (2.15) or (2.16) and (2.19), makes sense.

We shall characterize the fixed value $\bar{\mu}$ of the small parameter μ, which satisfies the conditions (2.10), (2.19). Let us consider, in the interval $[0, L\bar{\mu}^{-1}]$ the corresponding Cauchy problem (1.14) for the initial condition $y(t)|_{t=0} = y_0 \geqslant V[u_0, 0]$, where $V[u_0, 0] = (u_0(x), B(0) u_0(x))$, and the operator $B(0) = B(t)|_{t=0}$ according to (2.19).

Calculating the time-derivative from the functional $V[u, t]$ along the solution of problem (2.3)-(2.6) or, which amounts to the same, along the solution of problem (2.1), we obtain the inequality

$$dV[u(t, x), t]/dt \leqslant \lambda_{\max}(t) V[u(t, x), t], t \in [0, L\bar{\mu}^{-1}]$$

Then when $t \in [0, L\bar{\mu}^{-1}] \cap [0, L\mu^{-1}]$ along the solution $w(t, x)$ of the initial system the following system holds:

$$V[w(t, x), \partial w(t, x)/\partial t, t] \leqslant y_0 \exp[QL\mu_0^{-1}] \tag{2.20}$$
$$\mu_0^{-1} = \min\{\mu^{-1}, \bar{\mu}^{-1}\}, t \in [0, L\mu^{-1}] \cap [0, L\bar{\mu}^{-1}]$$

Therefore, by virtue of the theorem proved in Sect.1, the initial system (2.1) in the Hilbert space H with the above conditions on its parameters is technically stable in the finite interval of time (2.20) with respect to the measure $\rho(u, 0)$.

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